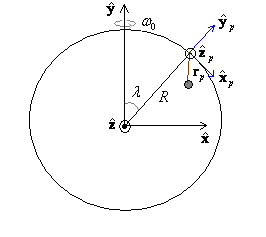
**N2L in Accelerated Reference Frames**

That was fun. Now going to do some harder ones. First we’ll do a problem from the inertial reference frame, just to see the inertial acceleration stuff come out independently of all our formalism.

**Example: Foucault Pendulum**

Let’s consider the oscillation of a pendulum from the perspective of the Earth’s surface.

Consider our Earth, and you at a certain angle, λ, from the top of its axis of rotation,



What would be the equations of motion of this object in your reference frame? Well, let **R** point from the center of the Earth to your coordinate system and then let **r**p point from your coordinate system to the particle/pendulum. For simplicity, we’ll place the origin of our coordinate system at the origin of the pendulum’s string. And let the length of the pendulum be ℓ. As before then, we have:



Now let’s just keep the first order terms in ω0 – the term second order in ω0 is much smaller and can be expected to be negligible in the case of rotations on the surface of the Earth. This means that we are neglecting the fictitious inertial force and centrifugal force and keeping just the Coriolis force. So this brings us to…

.

and working out the cross product…



So then splitting it up into its components we get:



We might wonder why we have 4 unknowns and 3 equations, which would apparently make this equation unsolvable. But there is another condition we have, namely Ɩ = sqrt(x2 + y2 + z2). So really have 4 equations and 4 unknowns.

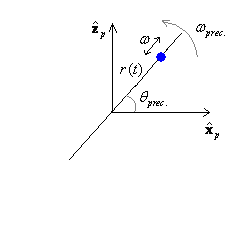
We’re interested in the precessional motion of the pendulum, which is governed by the x and z coordinates. And we will make another approximation to simplify our equations – we just want to see what the rough precessional period of the pendulum is. So we will assume the pendulum is precessing at a roughly constant height yp above the ground in our reference frame. So we will take dy/dt ≈ 0. Also, we will take the tension force in the rope to be nearly equal to gravity, so T ≈ mg. So then we would get for the x and z equations…



and,



We’ll look for periodic steady state solutions of a particular form. We’re looking for solutions that describe a pendulum oscillating back and forth along a certain axis, which also precesses at a certain rate – illustrated below:



Consider a particle fixed at radius r along an axis which precesses at rate ωprec. Then the position as a function of time would be:



But now allow the radius to vary with time – as the particle oscillates like a pendulum up and down the axis. Suppose the rate of oscillation is ω. Then r(t) = rcos(ωt), assuming it starts at r at time t = 0. So now we have:



So the primed coordinates as a function of time ought to be:



Let’s look for solutions to our ODE with this form. So taking the necessary derivatives:



And filling these into our ODE we get:



Simplifying…



Now in order for this equation to be 0 for all times, we must have the quantities in brackets be equal to zero themselves. So from the first equation we must have:



and the second equation reiterates the first. Let’s solve these two equations for ω – the oscillation frequency along the axis, and ωprec. – the precessional frequency of that axis. Upon dividing out the ω, the bottom equation can be solved for ωprec. and we get:



Then filling this into the top equation we get:



So altogether we have:



Note direction of precession is clockwise in the northern hemisphere and counter clockwise in the southern hemisphere, consistent with what one expects from the Coriolis force (so did I get a minus sign wrong – cause r(t) formula suggests CCW precession). Remember that ω0 is the rate of rotation of the Earth about its axis, and is much smaller than g/ℓ, the rotational period of the pendulum along its axis. So the correction due to the rotation of the Earth to *this* quantity is negligible. But the other major effect we see is that the axis along which the pendulum rotates precesses at a rate cosλ times the rate at which the Earth rotates. So at the North pole where λ = 0o, we would find that the precessional frequency matches the Earth’s rotation frequency. And at the equator, where λ = 90o, we would find no precession of the axis along which the pendulum rotates.

Just in case you don’t like guessing the solution to ODE’s, but want to derive them instead I’ll point out that if we want to look for a steady state rotational solution to our 2 coupled ODE’s, then we could postulate a solution form of:



where X and Z are just some numbers. Then filling these into our equation, and leaving ‘taking the real part’ implicit, we would have:



and therefore:



which has the matrix form,



and only has a non-zero solution if:



which implies a frequency (squared) of oscillation (using quadratic formula on ω2):



But we need ω. Luckily the square root of all of this doesn’t turn out so bad.



and so the square root is:



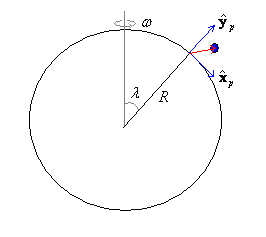
where we recognize the first term as the precessional frequency, and the second as the pendulum’s frequency of oscillation along its axis. And so there are 4 values of ω, and so 4 possible solutions. So the general solution to our equation is a linear combination of each possible solution.



But we won’t mess with this any further since I want to go to Zaxby’s and get some wings.

**Example: projectile motion general equations, including only Coriolis force**

Now let’s take into account Earth’s rotation and see how this changes the trajectory of a particle, if much at all. Consider our Earth, and you at a certain angle from the top of its axis of rotation,



What would be the equations of motion of an object in your reference frame? We have:



Now let’s just keep the first order terms in ω (so basically, just keeping the Coriolis force). This brings us to…

.

and working out the cross product…



So then splitting it up into its components we get:



Now let’s solve for the coordinates. The Laplace transform is kind of unwieldy at this point. So let’s integrate w/r to t on the first two:



And now plug the top two into the z equation…



Where in the last line we try to be consistent and ignore ω2 order terms…So then two integrations gives,



and now we can get



and so,



ignoring higher order ω terms again. Finally y will be given by:



and integrating we get:



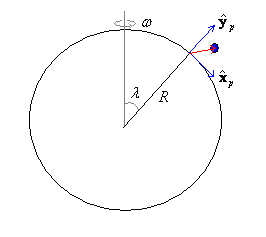
So altogether we have:



Clearly if we ignore the rotation of the Earth, then we just get the usual suspects. But otherwise…

**Example: Dropping ball from building**

If we drop a ball from rest at a height h, how long will it take to hit ground? What will be the deflection of a ball? So going back to our drawing:



and our y-equation.



To get the time we have:

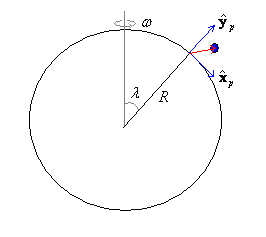


The deflection in the z direction will be:



Thus the drift is in the same direction as the rotation of the Earth. This is at first unintuitive, but makes sense given that the ball is at a higher point than the surface of the Earth and so is moving slightly faster w/r to the center of the Earth than the surface is. So when released, it will continue moving at that speed, outpacing the surface a little, and landing in front of the building. The direction of the force also follows from the direction of the Coriolis force, which is Fc = -2m**ω×vp**, where vp is predominantly in the negative yp direction. Finally from the x-equation, the deflection in the x direction would be 0.

**Example**  
Imagine a hole is bored through the center of the Earth, located at the equator. You perform an experiment where you drop a small steel ball down the hole. Surprisingly, every time you repeat this experiment, the ball eventually hits the side of the hole. After how many seconds does this impact occur if your hole has a 1m radius and you drop the ball from the exact center? Ignore air resistance. Again, we go back to:



Now λ = π/2. And we want the horizontal coordinates: x, and z. These are:



Filling in the angle, and initial position, velocity,



So it will hit the edge when (setting z = -1),



And note z should be negative, as basically the horizontal velocity of the ball, at the surface, in the z-direction, is faster than that horizontal velocity the Earth has at points with lower elevation, and so the ball will overshoot, in effect, the points at lower elevations. Let’s redo the problem from the very beginning,



So our equations are:



To first order in Ω, we have:



We can integrate the y equation,



and plug into the z-equation,



We can ignore Ω2 terms again,



Two integrations gives,



Filling in z0 = 0, 0 =0, and 0 = 0, we have:

